Theory of $4e$ versus $2e$ supercurrent in frustrated Josepshon-junction rhombi chain

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Abstract

We consider a chain of Josepshon-junction rhombi (proposed originally in [1]) in quantum regime, and in the realistic case when charging effects are determined by junction capacitances. In the maximally frustrated case when magnetic flux through each rhombi Φ_r is equal to one half of superconductive flux quantum Φ_0, Josepshon current is due to correlated transport of *pairs of Cooper pairs*, i.e. charge is quantized in units of $4e$. Sufficiently strong deviation $\delta\Phi \equiv |\Phi_r - \Phi_0/2| > \delta\Phi^c$ from the maximally frustrated point brings the system back to usual $2e$-quantized supercurrent. We present detailed analysis of Josepshon current in the fluctuation-dominated regime (sufficiently long chains) as function of the chain length, E_J/E_C ratio and flux deviation $\delta\Phi$. We provide estimates for the set of parameters optimized for the observation of $4e$-supercurrent.

I. INTRODUCTION

Pairing of Cooper pairs in frustrated Josephson junction arrays was theoretically proposed recently [1, 2, 3] in the search of topologically protected nontrivial quantum liquid states. The simplest system where such a phenomenon could be observed was proposed by Doucot and Vidal in [1]. It consists of a chain of rhombi (each of them being small ring of 4 superconductive islands connected by 4 Josephson junctions) placed into transverse magnetic field. cf. Fig. 1. It was shown in [1] that in the fully frustrated case (i.e. magnetic flux through each rhombus $\Phi = \frac{1}{2}\Phi_0 = \frac{hc}{4e}$) usual tunnelling of Cooper pairs along the chain is blocked due to destructive inteference of tunneling going through two paths within the same rhombus, while correlated 2-Cooper-pair transport survives. Evidently, experimental observation of such a phenomenon (detected as anomalous period $\frac{1}{2}\Phi_0$ of the global supercurrent along the chain) would be very desirable. However, theoretical results of Ref. [1] refer to the situation when Coulomb energy is determined by self-capacitances C_0 of individual superconductive islands, whereas in real submicron Josephson-junction arrays capacitances of *junctions* C dominate, cf. e.g. [4]. In this paper we reconsider the model of Ref. [1] for the experimentally relevant situation $C \gg C_0$. This case is also simpler for theoretical treatment, since Lagragian of the system becomes a sum of terms, such that each of them belongs to individual rhombus only. The only source of coupling between different rhombi is the periodic boundary condition along the chain. The method to treat similar problem was developed recently by Matveev, Larkin and Glazman [5] (MLG). They considered simple chain of N Josephson junctions in the closed-ring geometry, and reduced calculation of supercurrent in large-N limit to the solution of a Schröedinger equation for a particle moving in a periodic potential $\sim \cos x$, with appropriate boundary condition. MLG assumed (we will do the same) that Josephson energy E_J of junctions is large compared to their charging energy $E_C = e^2/2C$. We will generalize the MLG method in order to use it for the case of ring of rhombi. It will be shown that in our case fictitios particle of the MLG theory is still moving in the cos-like potential, but it acquires now large *spin* $S = \frac{1}{2}N$, where N is the number of rhombi in the ring. In the maximally frustrated case $|\Phi_r - \Phi_0/2| \equiv \delta\Phi = 0$ the x-projection of the spin is an integral of motion, which should be chosen to minimize the total energy. As a result, $S_x = \pm\frac{1}{2}N$ and the whole problem reduces to the one studied by MLG up to trivial redefinition of parameters. In this situation ground-state energy and

supercurrent (which is proportional to derivative of the ground-state energy over total flux Φ_c) are periodic function of Φ_c with period $\Phi_0/2$, i.e. $4e$-transport takes place. Nonzero flux deviation $\delta\Phi$ produces longitudinal field h_z coupled to the z-component of spin of fictitios particle, which acquires now nontrivial dynamics. We show that in the limit of sufficiently long rhombi chain the whole problem can be analyzed in terms of semiclassical dynamics of a particle with a large spin under spin-dependent potential barrier. In general, there are two tunnelling trajectories, one of them corresponds to usual $2e$ transport, whereas another to $4e$ transport. Comparing actions of these trajectories for different $\delta\Phi$, we find critical flux deflection $\delta\Phi_c$ as function of the ratio $E_J/E_C \gg 1$.

The rest of the paper is organized as follows: in Sec. II we define our model and identify its classical states; in Sec. III we derive effective Hamiltonian which governs quantum phase slip processes and calculate supercurrent as function of the flux deflection $\delta\Phi_c$; in Sec. IV we consider current-bias chain with $I > I_c$ and calculate voltage $V(I)$ via the rate of incoherent quantum phase slips. Our conclusions and suggestions for the experiment are presented in Sec. V. Finally, in Appendix A somewhat tedious calculation of current-phase relation is presented.

II. THE MODEL AND ITS CLASSICAL STATES

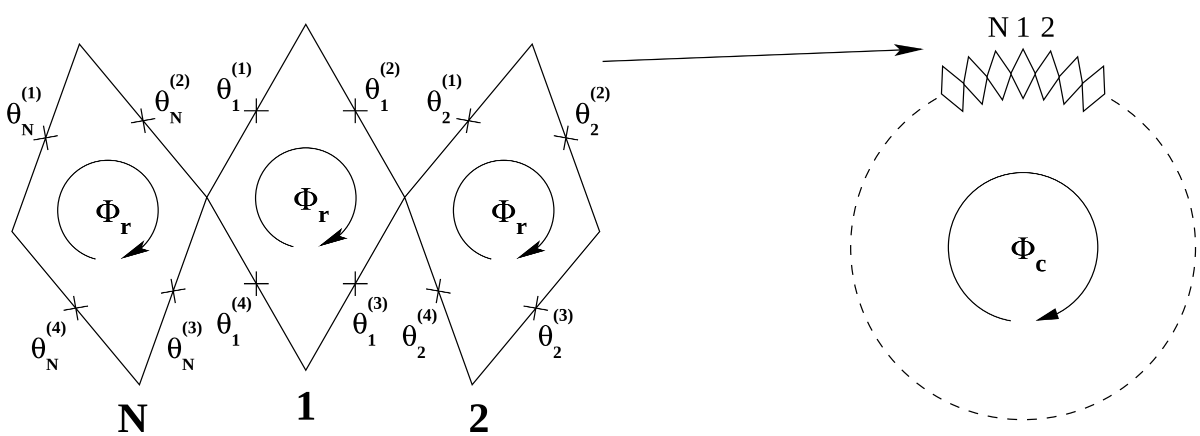

Figure 1: The chain of rhombi as a closed ring.

We study a chain of N rhombi shown in Fig. 1. Each rhombi consists of four superconductive islands connected by tunnel junctions with Josephson coupling energy $E_J = \hbar I_c^0/2e$; charging energy E_C is determined by capacitance C of junctions, $E_C = e^2/2C$ (we neglect

self-capacitances of islands which are assumed to be much smaller than C). Below we consider Josephson current along the chain of $N \gg 1$ rhombi and assume that the chain is of the ring shape, with total magnetic flux Φ_c inside the ring. We also denote by Φ_r the flux per elementary rhombus plaquette and define phases γ and φ:

$$\gamma = 2\pi \frac{\Phi_c}{\Phi_0}\,, \qquad \varphi = 2\pi \frac{\Phi_r}{\Phi_0}\,, \tag{1}$$

where $\Phi_0 = h/2e$ is the superconducting flux quantum. We study the situation than Φ is close to $\Phi_0/2$, i.e. $\delta = \varphi - \pi \ll 1$.

Assuming that the charge transport through the system at $\delta = 0$ is carried out by charges $4e$, we expect that in this case dependence of the current in the chain on the external flux Φ_c is periodic with period $\Phi_0/2$. Below we calculate the $\Phi_0/2$-periodic current at $\delta = 0$. We also show that at small δ the current through the system has two components I_{4e} and I_{2e} with periods $\Phi_0/2$ and Φ_0 respectively. The first component corresponds to the current carried by pairs of Cooper pairs and the second one corresponds to the single Cooper pair transport. At very small δ the current I_{4e} dominates over I_{2e}. We will refer to this regime as to $4e$-regime. At large enough δ the opposite situation ($2e$-regime) is realized. We will find below the crossover point δ_c between these two regimes.

The system is described by the following imaginary-time action

$$S_E = \int dt \sum_{n=1}^{N} \sum_{m=1}^{4} \left\{ \frac{1}{16E_C} \left(\frac{d\theta_n^{(m)}}{dt} \right)^2 - E_J \cos \theta_n^{(m)} \right\}. \tag{2}$$

Here the variable $\theta_n^{(m)}$ is the phase difference across the m-th junction in the n-th rhombus (see Fig. 1). Taking into account that each rhombus is pierced by flux Φ_r and the flux through the whole chain is Φ_c we derive the following additional conditions

$$\sum_{m=1}^{4} \theta_n^{(m)} = \varphi\,, \qquad n = 1, 2, .., N\,, \tag{3}$$

$$\sum_{n=1}^{N} \left(-\theta_n^{(3)} - \theta_n^{(4)} \right) = \gamma\,. \tag{4}$$

In this paper we consider the case of strong coupling between grains $E_J \gg E_C$. This enables us to use semi-classical approximation for calculating the energy spectrum of the system. At $E_C = 0$ the phases $\theta_n^{(m)}$ become classical variables and the energy states of the

chain can be found by minimizing the sum of Josephson energy terms in action (2). Let us introduce variables $\theta_n = -\theta_n^{(3)} - \theta_n^{(4)}$, where θ_n is the phase difference along the diagonal of the n-th rhombus. It is convenient to make minimization in two steps. First of all the Josephson energy of a single rhombus under the fixed flux through the rhombus and under the fixed phase difference θ_n is minimized. For the Josephson energy of the chain we then get for $\delta \ll 1$:

$$E = -2\sqrt{2}E_J \sum_{n=1}^{N} \left(1 + \frac{1}{4}\delta\sigma_n^z\right) \cos\left(\frac{\theta_n}{2} - \beta_n\right) , \tag{5}$$

$$\sin\beta_n = \pm\frac{\sigma_n^z}{\sqrt{2}}\left(1 - \frac{\delta}{4}\sigma_n^z\right) , \qquad \cos\beta_n = \pm\frac{1}{\sqrt{2}}\left(1 + \frac{\delta}{4}\sigma_n^z\right) . \tag{6}$$

Plus and minus signs in (6) correspond to positive (resp. negative) values of $\cos\frac{\theta}{2}$. Here we have introduced an important notation $\sigma_n^z = \mathrm{sign}\,\sin\theta_n$. It can be easily shown that at $\delta = 0$ each individual rhombus has two classical ground states with equal energies. This states differ only in the sign of the superconducting current circulating around the plaquette which corresponds to the binary variable σ_n^z.

Now we have to minimize the energy (5) with respect to phases $\theta_n^{(m)}$ subject to the constrains (4). Assuming δ to be small and N to be large we get

$$E_{m,\sigma} \approx \frac{E_J\sqrt{2}}{4N}(\widetilde{\gamma} - \pi N/2 - \pi S^z - 2\pi m)^2 - \sqrt{2}\delta S^z E_J + \mathrm{Const.} \tag{7}$$

Here m is an arbitrary integer (which has the same meaning as in the MLG paper [5]) and

$$s_n^z = \frac{1}{2}\sigma_n^z , \qquad S^z = \frac{1}{2}\sum_{n=1}^{N}\sigma_n^z , \qquad \widetilde{\gamma} = \gamma + \frac{N\varphi}{2} = \gamma + \frac{\pi N}{2} + \frac{N\delta}{2} . \tag{8}$$

In the above equation s_n^z can be considered as z-projection of the "spin" $\frac{1}{2}$ which describes binary degeneracy of states of the n-th rhombi. Then S_z corresponds to the z-projection of the total large spin $\mathbf{S}$ describing the whole rhombi chain. For clarity everywhere in this paper we will refer to the case of even number of rhombi. Then total spin S and eigenvalues of its projection S^z are integer. Classical states of the chain are characterized by individual spin projections σ_n^z for each rhombus, and by collective integer-valued variable m. We will denote these states by $|m, \{\sigma_n^z\}\rangle$ or $|m, \sigma >$. Physically, classical state of the chain is characterized by the global current I along the chain, and by the signs of local currents flowing in each of N rhombi. Nonzero charging energy E_C provides quantum phase slips in each of $4N$ Josephson junction; these processes mix different classical states leading to formation of the

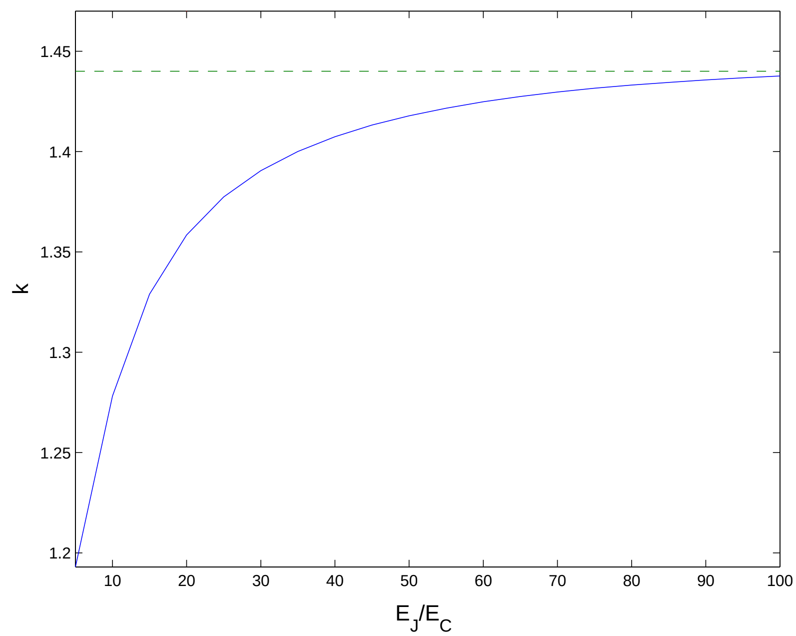

Figure 2: The numerical factor k as a function of E_j/E_c.

fully quantum ground state. Below we derive effective Hamiltonian acting on the space of classical states, and find ground-state energy $E_0(\gamma)$ and corresponding supercurrent.

III. QUANTUM FLUCTUATIONS OF RHOMBI AND SUPERCURRENT

We turn to analysis of quantum fluctuations of $\theta_n^{(m)}$ at finite E_C. The most important type of these fluctuations involve an instanton (quantum phase slip, QPS), i.e. a trajectory that begin at one of the minima (7) of the potential energy in action (2) at $t = -\infty$ and ends at another minima at $t = +\infty$. There are two kinds of trajectories: the first starts at $|m, \{\sigma_n^z\}\rangle$ and ends at $|m, \{\sigma_n^z + 2\delta_{nk}\}\rangle$ for arbitrary $1 \leq k \leq N$, $\sigma_k^z = -1$, whereas the second start at $|m, \{\sigma_n^z\}\rangle$ and end at $|m+1, \{\sigma_n^z - 2\delta_{nk}\}\rangle$ for arbitrary $1 \leq k \leq N$, $\sigma_k^z = 1$. Any trajectory of the first kind corresponds to QPS in $\theta_k^{(1)}$ or $\theta_k^{(2)}$, whereas trajectory of the second kind corresponds to QPS $\theta_k^{(3)}$ or $\theta_k^{(4)}$. Note that at $\delta = 0$ and $\gamma = \pi/2$ all these trajectories starting at $|m, \{\sigma_n^z\}\rangle$ with $2m + S^z = 0$ connect the minima with equal energies. Thus they are important for restoring symmetry of the system which is classically broken. Let us denote as v the amplitude of a QPS in one contact. At large $N \gg 1$ this amplitude does not differ from the "spin flip" amplitude for a single rhombus at $\Phi_r \approx \Phi_0/2$. In this

approximation we can use result from Ref. [2]:

$$\upsilon \approx k \left(E_J^3 E_C\right)^{1/4} \exp\left(-1.61\sqrt{\frac{E_J}{E_C}}\right). \tag{9}$$

where k is a numerical factor of the order of unity. Comparison with direct numerical diagonalization [6] of of low-lying spectrum of a single frustrated rhombi shows that coefficient k grows from approximately 1.3 to 1.44 as the ratio E_J/E_C varies from 10 to infinity, cf. Fig. 2. The instantons account for the possibility of the system tunneling between different minima (7) of the potential energy in action (2). The effect of the instantons on the ground state energy can be represented by a tight-binding Hamiltonian defined as

$$\begin{aligned}\widehat{H}\left|m,\{\sigma_n^z\}\right\rangle = E_{m,\sigma}\left|m,\{\sigma_n^z\}\right\rangle + \\ 2\upsilon\sum_{k=1}^{N}\left|m,\{\sigma_1^z\,,\cdots\sigma_{k-1}^z\,,-\sigma_k^z\,,\cdots\sigma_{k+1}^z\,,\sigma_N^z\}\right\rangle + \\ 2\upsilon\sum_{k=1\,,\,\sigma_k^z=1}^{N}\left|m+1,\{\sigma_1^z\,,\cdots\sigma_{k-1}^z\,,-\sigma_k^z\,,\cdots\sigma_{k+1}^z\,,\sigma_N^z\}\right\rangle + \\ 2\upsilon\sum_{k=1\,,\,\sigma_k^z=-1}^{N}\left|m-1,\{\sigma_1^z\,,\cdots\sigma_{k-1}^z\,,-\sigma_k^z\,,\cdots\sigma_{k+1}^z\,,\sigma_N^z\}\right\rangle\,.\end{aligned} \tag{10}$$

To find the ground state energy $E(\gamma)$ it is convenient to make Fourier transformation over variable m according to

$$|x,\sigma\rangle = \sum_m \exp\left\{2i\left(2m - \frac{\widetilde{\gamma}}{\pi} + S^z + \frac{N}{2}\right)x\right\}|m,\sigma\rangle\,, \quad S^z = \sum_{n=1}^{N} s_n^z. \tag{11}$$

Note that not all vectors of our new basis (11) are independent. It is easy to see from (11) that $|x+\pi/2,\sigma\rangle = e^{-i\widetilde{\gamma}+i\pi S^z+i\pi N/2}|x,\sigma\rangle$. Considering any system's state of the form $|\psi\rangle = \sum_{x,\sigma}\psi(x,\sigma)\,|x,\sigma\rangle$ we should impose on the wavefunction $\psi(x,\sigma)$ a twisted boundary condition

$$e^{i\pi\widehat{S}^z+i\pi N/2}\psi\left(x+\pi/2,\sigma\right) = e^{i\widetilde{\gamma}}\psi(x,\sigma)\,. \tag{12}$$

Here we have introduced operator $\widehat{S}^z$ acting on the spin variables of the system according to standart rules.

The resulting Schrödinger equation acquires the form

$$\frac{\partial^2\psi}{\partial x^2} + (b - 2w\cos 2x\cdot\widehat{S}^x + 2h\widehat{S}^z)\psi = 0\,, \tag{13}$$

where

$$b = \frac{16NE}{\sqrt{2}E_J\pi^2}\,, \qquad w = \frac{64N\upsilon}{\sqrt{2}E_J\pi^2}\,, \qquad h = \frac{8N\delta}{\pi^2}\,. \tag{14}$$

Note that symmetry group of the Hamiltonian corresponding to the equation (13) includes transformations $U_n = e^{i\pi n\left(\widehat{S}^z+N/2\right)}\widehat{T}_{\pi n/2}$. Here $\widehat{T}_a$ is operator of the translation over distance a along the x-axis. Equation (12) shows that the parameter $\widetilde{\gamma}$ specifies different irreducible representations of the symmetry group.

The equations (13,12) allow comprehensive analytical investigation for the case when the flux per single rhombus equals $\Phi_0/2$. For such a system $h = 0$ and the Hamiltonian commutes with S^x. However, variables x and S are still coupled due to boundary condition (12). Therefore we look for wavefunction in the form

$$\psi(x,\sigma) = e^{i\widetilde{\gamma}}\,|S^x\rangle\,\phi(x) + e^{i\pi\widehat{S}^z+i\pi N/2}\,|S^x\rangle\,\phi(x+\pi/2) \tag{15}$$

Then for $\phi(x)$ we get standard Mathieu equation:

$$\frac{\partial^2\phi}{\partial x^2} + (b - 2q\cos 2x)\phi = 0\,, \tag{16}$$

where $q = wS^x$. Boundary condition (12) now reads as follows:

$$\phi(x+\pi) = e^{2i\widetilde{\gamma}}\phi(x)\,. \tag{17}$$

The ground state of the system defined by Eqs. (16, 17) corresponds to maximal absolute value of S_x, equal to $N/2$. In other words, in the ground state all rhombi in the chain are either in symmetric superpositions or in antisymmetric superpositions, of their double-degenerate classical states. Thus there are two degenerate eigenstates

$$|0^+_{\widetilde{\gamma}}\rangle = \phi_{\widetilde{\gamma}}(x)|S^x = N/2\rangle \qquad \text{and} \qquad |0^-_{\widetilde{\gamma}}\rangle = \phi_{\widetilde{\gamma}}(x+\pi/2)|S^x = -N/2\rangle = \widehat{U}_1|0^+_{\widetilde{\gamma}}\rangle\,, \tag{18}$$

with the same lowest energy E_0. This degeneracy is a direct consequence of the fact that for $h = 0$ (fully frustrated chain) the Hamiltonian has two symmetry operators S^x and U_1 which do not commute with each other (thus double-degeneracy refers to all states, not only to the ground-state). Eigenstates $|0^\pm_{\widetilde{\gamma}}\rangle$ constitute a basis where $\widehat{S}^x$ operator is diagonal. Coming back to the original problem defined by Eqs.(12, 13) we note, that in accordance with (15) the correct (unique) eigenstate obeying Eq.(12) can be constructed as specific linear combination of $|0^+_{\widetilde{\gamma}}\rangle$ and $|0^-_{\widetilde{\gamma}}\rangle$:

$$|G_{\widetilde{\gamma}}\rangle = \frac{e^{i\widetilde{\gamma}}|0^+_{\widetilde{\gamma}}\rangle + |0^-_{\widetilde{\gamma}}\rangle}{\sqrt{2}} \tag{19}$$

which diagonalize operator U_1. The eigenstate $|G_{\widetilde{\gamma}}\rangle$ is similar to the eigenstates $|G\rangle$ of Ref. [2], cf. Eq.(5) of that paper.

It follows from the boundary condition (17) that shift of the phase $\widetilde{\gamma}$ by π does not change the boundary problem defined by Eqs. (16, 17). Thus the ground-state energy of the system and the supercurrent through the circuit are periodic functions of the flux Φ_c with period $\Phi_0/2$.

At $Nw \sim N^2 \upsilon/E_J \ll 1$ fluctuations are weak, the amplitude of potential energy in (16) is small and its effect is most significant when $4\Phi_c/\Phi_0$ is integer and the energy levels $E_{m,\sigma}$ are degenerate. In this regime the usual approximation for semiclassical weak link is valid, and for the persistent current through the circuit we obtain

$$I(\widetilde{\gamma}) = \frac{2e}{\hbar}\frac{dE}{d\gamma} = \operatorname{sign}\widetilde{\gamma}\frac{\sqrt{2}I_c^0\pi}{4N}\left(1 - \frac{2|\widetilde{\gamma}|}{\pi}\right)\left(\frac{1}{\sqrt{(1-2|\widetilde{\gamma}|/\pi)^2 + (q/2)^2}} - 1\right). \qquad (20)$$

Phase-dependent current $I(\widetilde{\gamma})$ is described by equation (20) for $-\pi/2 < \widetilde{\gamma} < \pi/2$ and is a periodic function of $\widetilde{\gamma}$ with period π. So in the regime of weak fluctuations the dependence $I(\widetilde{\gamma})$ demonstrates sawtooth behavior slightly rounded due to fluctuations.

The opposite limit $Nw \gg 1$ corresponds to the regime of strong fluctuations. In this case the dependence of the eigenvalue b on the phase $\widetilde{\gamma}$ is exponentially weak [7]:

$$b = -2q + 16\sqrt{\frac{2}{\pi}}q^{3/4}e^{-4\sqrt{q}}(1 - \cos 2\widetilde{\gamma}) \qquad (21)$$

and for the persistent current in the ground state we find

$$I(\widetilde{\gamma}) = 32 \cdot 2^{3/8} I_c^0 \left(\upsilon/E_J\right)^{3/4}\sqrt{N}\exp\left\{-\frac{16 \cdot 2^{1/4}}{\pi}N\sqrt{\frac{\upsilon}{E_J}}\right\}\sin 2\widetilde{\gamma}\,. \qquad (22)$$

Equation (22), together with Eq.(9), presents one of our main results: it gives the amplitude of $4e$ - periodic Josepshson current in the regime of maximal frustration.

Let us now turn to the investigation of the general situation described by equations (13) and (12). As was mentioned above if the flux per elementary plaquette differs slightly from half supercondicting flux quantum the persistent current through the chain has two components I_{4e} and I_{2e}. In the regime of strong fluctuations both these currents are exponentially small. The main exponential factors in the expressions for them can be found on the basis of equation (13) using the semi-classical approximation.

Note that equation (13) corresponds to a particle of mass 1 with spin S moving in one-dimensional potential

$$U(x,\vec{S}) = w\cos 2x\cdot S^x - hS^z\,, \tag{23}$$

the particle energy being $E^0 = b/2$. So denoting by θ and ϕ the angels determining the spin direction, we can write the imaginary time tunneling amplitude in the form of path integral [8]

$$\begin{aligned}&\langle\theta_2\,,\phi_2\,,x_2|\,e^{-T\widehat{H}}\,|\theta_1\,,\phi_2\,,x_1\rangle = \\ &\qquad\int_{\theta_1,\,\phi_1,\,x_1}^{\theta_2,\,\phi_2,\,x_2}\mathcal{D}\Omega\mathcal{D}x\exp\left\{-\int_{-T/2}^{T/2}d\tau\left(iS(1-\cos\theta)\dot{\phi}+\frac{\dot{x}^2}{2}+U(x\,,\vec{S})\right)\right\}\end{aligned} \tag{24}$$

For our future purposes it is more convenient to use the above path integral in another form also derived in [8]:

$$\begin{aligned}&\left\langle\vec{S}_2\,,x_2\right|e^{-T\widehat{H}}\left|\vec{S}_1\,,x_1\right\rangle = \\ &\qquad\int_{\vec{S}_1,\,x_1}^{\vec{S}_2,\,x_2}\mathcal{D}\vec{S}\mathcal{D}x\,\delta\left(\vec{S}^2-S^2\right)\exp\left\{-\int_{-T/2}^{T/2}d\tau\left(i\frac{S^x\dot{S}^y-\dot{S}^xS^y}{S+S^z}+\frac{\dot{x}^2}{2}+U(x\,,\vec{S})\right)\right\}\end{aligned} \tag{25}$$

We will analyse the expression (25) for the limit of relatively large δ when $h\gg w$. In this regime the field h in (23) fixes the direction of $\vec{S}$ so that S^x and S^y are always small. Therefore we can linearize the action in (24) with respect to S^x and S^y. After linearization we can easily exclude the variable S^y using the equations of motion. The substitution of variables $S^x\longrightarrow\sqrt{Sh}y$, $\tau\longrightarrow\tau/h$ leads to the path integral

$$\langle y_2\,,x_2|\,e^{-T\widehat{H}}\,|y_1\,,x_1\rangle = \int_{y_1,\,x_1}^{y_2,\,x_2}\mathcal{D}x\mathcal{D}y\,exp\,(-S_E) \tag{26}$$

where the action

$$S_E = \frac{h}{2}\int_{-\infty}^{+\infty}d\tau\left\{\dot{x}^2+\dot{y}^2+U_{eff}(x,y)\right\}\,, \tag{27}$$

$$U_{eff}(x,y) = (y+d\cos 2x)^2+d^2\sin^2 2x\,,\qquad d=\sqrt{\frac{w^2S}{h^3}}=\frac{\sqrt{2}\pi}{\delta^{3/2}}\frac{\upsilon}{E_J}\,. \tag{28}$$

The appropriate equations of motion are

$$\ddot{x}+2dy\sin 2x = 0\,, \tag{29}$$

$$\ddot{y}-y-d\cos 2x = 0\,. \tag{30}$$

Using semi-classical approximation we should first determine the classical minima of the potential (28). Within the same limit $h \gg w$ we find that U_{eff} has two groups of minima (we call them "even" and "odd" minima)

$$x = \pi n \, , \qquad y = -d \, , \tag{31}$$

$$x = \frac{\pi}{2} + \pi n \, , \qquad y = d \, , \tag{32}$$

where n is an arbitrary integer. All these minima correspond to the same value of $U_{eff} = 0$. So we have to consider two types of tunnelling trajectories. Trajectories of the first type connect minima of the same group, i.e. "even-even" and "odd-odd", and corresponding variation of the variable x between minima is $\pm\pi$, whereas y returns to its original value. Trajectories of the second type connect minima of opposite parity (i.e. opposite signs of y), and change x variable by $\pm\frac{\pi}{2}$. It is not difficult to see from Eqs.(11,13,12), that increment Δx of the variable x along tunnelling trajectory is in one-to-one correspondence to the elementary charge transported along the rhombi chain: $q_0 = \frac{4e}{\pi}\Delta x$. Therefore trajectories of the first type lead to $4e$ - supercurrent, whereas trajectories of the second type produces usual $2e$-quantized supercurrent. The amplitudes of the supercurrent components are determined (cf. Appendix for the direct derivation) primarily by the classical actions on corresponding trajectories:

$$I(\gamma) = I_{2e} \sin \widetilde{\gamma} + I_{4e} \sin(2\widetilde{\gamma}), \tag{33}$$

where

$$I_{4e} = A_{4e} \exp(-S_E^{4e}), \quad I_{2e} = A_{2e} \exp(-S_E^{2e}) \, , \tag{34}$$

and S_E^{4e} and S_E^{2e} are the values of tunnelling actions on trajectories of the first and second type respectively. Both S_E^{4e} and S_E^{2e} are large in the region of strong fluctuations $Nw \gg 1$, thus the total supercurrent will in general be dominated by least-action processes.

To compare actions S_E^{4e} and S_E^{2e} we note that the dynamical system (29) and (30) has two characteristic frequencies. The first one characterises "spin" subsystem with $\omega_s = 1$, whereas the second one is the frequency of the "x" subsystem, $\omega_x \sim d$, since typical value of y in (29) is d. Therefore at $d \ll 1$ - i.e. at sufficiently large flux deflections δ, the "spin variable" y is fast and can be integrated out in adiabatic approximation, which leads to

$$S_E = h \int d\tau \left\{ \frac{\dot{x}^2}{2} - \frac{d^2}{4} \cos 4x \right\} \, , \tag{35}$$

$$S_E^{4e} \approx 2hd, \quad \text{and} \quad S_E^{2e} \approx hd, \;\; \text{at} \quad d \ll 1 \tag{36}$$

The dominant process is thus usual $2e$ transfer. Comparing the action (35) with the action corresponding to the Shrödinger equation (16)

$$S_E^0 = \int d\tau \left\{ \frac{\dot{x}^2}{2} + q\cos 2x \right\} \tag{37}$$

and using (21) we obtain supercurrent amplitude

$$I_{2e} \approx 32 \cdot 2^{1/4} I_c^0 \sqrt{N} \left(\frac{\upsilon}{E_J\sqrt{\delta}} \right)^{3/2} \exp\left\{ -\frac{8\sqrt{2}}{\pi} \frac{N\upsilon}{E_J\sqrt{\delta}} \right\} \quad \text{at} \quad \frac{1}{\delta^{3/2}} \frac{\upsilon}{E_J} \ll 1 \tag{38}$$

At small flux deflection δ the parameter $d \gg 1$ and the spin variable y is relatively slow and almost does not change on the type-1 trajectory. The dominant trajectory is then $4e$-one. Assuming y to be constant, we get

$$S_E^{4e} = h \int d\tau \left\{ \frac{\dot{x}^2}{2} - d^2 \cos 2x \right\} = 4hd \tag{39}$$

Taking into account also the first-order term of perturbation theory over $1/d \ll 1$, we find $S_E^{4e} = h(4d-1)$. Comparison of the action (39) with (37) and (21) allows us to determine the pre-exponential factor in the expression for the current

$$I_{4e} \approx 128 \cdot 2^{1/4} I_c^0 \sqrt{N} \left(\frac{\upsilon}{E_J\sqrt{\delta}} \right)^{3/2} \exp\left\{ -\frac{32\sqrt{2}}{\pi} \frac{N\upsilon}{E_J\sqrt{\delta}} + \frac{8N\delta}{\pi^2} \right\} \quad \text{at} \quad \frac{1}{\delta^{3/2}} \frac{\upsilon}{E_J} \gg 1 \tag{40}$$

Note that at δ determined from the equation $h = w$ (where the linear approximation used to describe the spin degree of freedom fails), the $4e$-current from (40) matches the exact result for $\delta = 0$ presented in (22).

In the intermediate region of $d \sim 1$ we analyse equations (27), (29) and (30) numerically. First we write S_E^{4e} and S_E^{2e} in the form $S_E^{4e} = h\widetilde{S}_E^{4e}(d)$, $S_E^{2e} = h\widetilde{S}_E^{2e}(d)$. Then the functions $\widetilde{S}_E^{4e}(d)$ and $\widetilde{S}_E^{2e}(d)$ depending on a single parameter d have been evaluated numerically. The result is presented on Fig. 3. The actions for both types of trajectories are equal at $d = d_0 \approx 3.2$, where we have $\widetilde{S}_E^{4e}(d_0) = \widetilde{S}_E^{2e}(d_0) \approx 11.9$. Thus the crossover between $4e$-regime and $2e$-regime takes place at

$$\delta\Phi = \delta\Phi^c = \left(\frac{\upsilon^2}{4\pi d_0^2 E_J^2} \right)^{1/3} \Phi_0 \approx 0.2 \left(\frac{\upsilon}{E_J} \right)^{2/3} \Phi_0 \tag{41}$$

Varying flux Φ_r in some vicinity of crossover point (41) one can find both $2e$ and $4e$ components of supercurrent, but their relative weight is expected to vary strongly with $\Phi_r - \Phi_r^c$, in some analogy with phase coexistence near first-order phase transition.

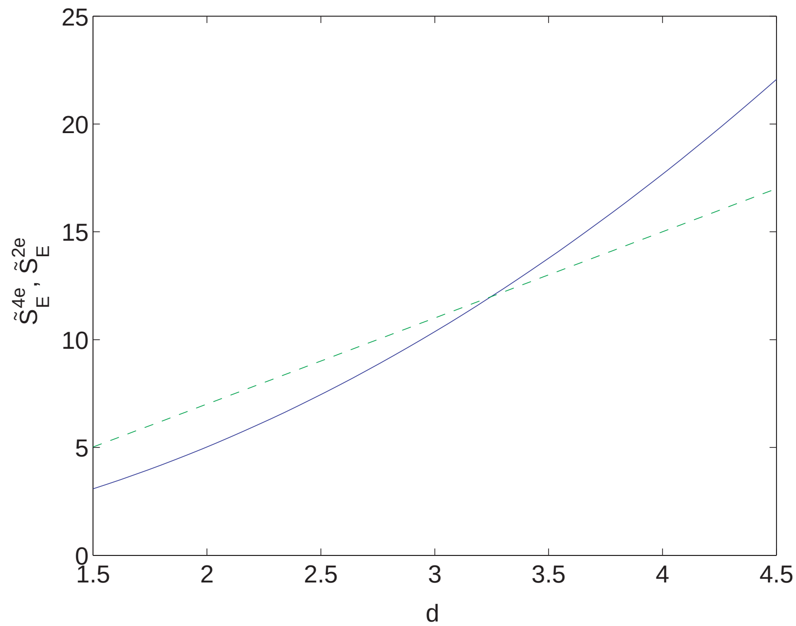

Figure 3: The results of numerical evaluation of $\widetilde{S}_E^{2e}(d)$ (solid line) and $\widetilde{S}_E^{4e}(d)$ (dashed line).

IV. LOW-VOLTAGE STATES

In previous Section we obtained estimates (20,22,38,40) for the equilibrium supercurrent $I(\gamma)$ around the *flux-biased* rhombi chain with $N \gg 1$. Note that the maximum value of this supercurrent is small, compared to individual critical current of a single junction I_c^0, even in the case of weak quantum fluctuations, cf. Eq.(20). This is due to the fact that in our analysis we have considered perfectly equilibrium Josepshson current, which must be 2π-periodic as function of the total phase bias γ. Therefore in the long chain phase differences across each rhombi scales as $1/N$, leading to $I_c \sim I_c^0/N$ in weak-fluctuation limit $Nw \ll 1$ (in the opposite limit I_c is small exponentially in N). It is clear, however, that under the condition of some *current bias*, with a fixed $I \ll I_c^0$, the chain will be in some "nearly superconducting" state with a very low voltage, due to rare phase slip processes. Below we consider regime of relatively large currents (the condition to be specified below), when the processes of tunnelling in different rhombi are incoherent. In this case mean voltage V along the whole chain can be estimated just as N times the voltage along a single rhombus. Below we estimate probability per unit time of an individual QPS in a single rhombus at the fixed transport current $I \ll I_c^0$, and find the $V(I)$ dependence.

Introducing variables $\theta = \theta^{(1)} + \theta^{(2)}$, $\chi_1 = \theta^{(1)} - \theta^{(2)}$, $\chi_2 = \theta^{(3)} - \theta^{(4)}$ we can rewrite the

imaginary-time action for a single rhombus carrying external current I in the form

$$S_E = \int d\tau \left\{ \frac{1}{32E_C} \left(2\dot{\theta}^2 + \dot{\chi_1}^2 + \dot{\chi_2}^2 \right) + V(\theta, \chi_1, \chi_2) \right\} , \tag{42}$$

$$V(\theta, \chi_1, \chi_2) = -E_J \left(2\cos\frac{\theta}{2}\cos\frac{\chi_1}{2} + 2\sin\frac{\theta}{2}\cos\frac{\chi_2}{2} + \frac{I}{I_c^0}\theta \right) . \tag{43}$$

We have assumed here that the flux inside the rhombus equals half the supercoducting flux quantum. In order to find the classical states of the rhombus we eliminate χ_1 and χ_2 from (43) and get

$$V(\theta) = -E_J \left(2\left|\cos\frac{\theta}{2}\right| + 2\left|\sin\frac{\theta}{2}\right| + \frac{I}{I_c^0}\theta \right) . \tag{44}$$

The potential (44) has a number of local minima $\theta_{min} = \theta_0 + \pi m$ where θ_0 is determined by the equation

$$\sin\frac{\theta_0}{2} - \cos\frac{\theta_0}{2} = \frac{I}{I_c^0} . \tag{45}$$

With an appropriate choice of phases χ_1 and χ_2 every θ_{min} corresponds to a classical state of the rhombus localized near this minimum. Due to quantum tunneling all these states are metastable and have finite decay time τ.

Within the semi-classical approximation (valid for $E_J \gg E_C$) the decay time τ is determined by vicinity of a bounce i.e. a classical trajectory starting at a minimum of the potential energy (43) coming close to another one and then going back to the first minimum [10, 11]. To be specific we will refer here to the decay rate of a state corresponding to $\chi_1 = 0$, $\chi_2 = 0$ and $\theta = \theta_0$. Decay of this state goes via one of two possible bounce trajectories (for $I > 0$). One of them passes near $\theta = \theta_0$, $\chi_1 = 2\pi$ and $\chi_2 = 0$ while the other passes near $\theta = \theta_0$, $\chi_1 = -2\pi$ and $\chi_2 = 0$. Both these bounces give equal contribution to the decay rate.

Let us denote by $q = (\theta, \chi_1, \chi_2)^T$ — the three-dimensional column-vector in the coordinate space of the rhombus. We also introduce $q_0(\tau) = (\theta_0, 0, 0)^T$ as the trajectory corresponding to the system being at the minimum of the potential (43) and $q_b(\tau)$ as the bounce trajectory which can be determined by solving the classical equations of motion. The decay probability per unit time of the unstable state is given by [10, 11]

$$1/\tau = 2\left(\frac{S_E[q_b]}{2\pi}\right)^{1/2} e^{-S_E[q_b]} \left| Det'\left(\frac{\delta^2 S_E}{\delta q^2}\right)_{q=q_b} \right|^{-1/2} \left| Det\left(\frac{\delta^2 S_E}{\delta q^2}\right)_{q=q_0} \right|^{1/2} \tag{46}$$

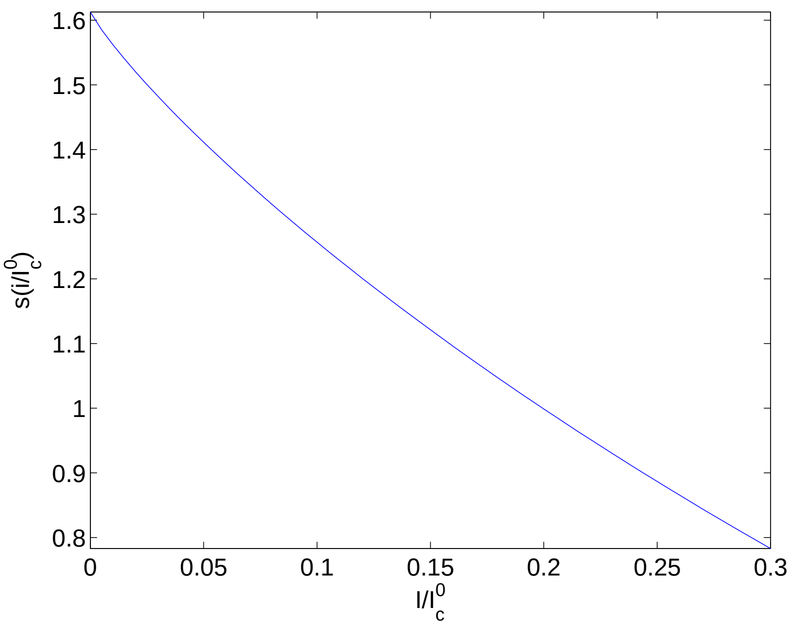

Figure 4: Results of numerical evaluation of $s(I/I_c^0)$.

where Det' indicates that the zero eigenvalue is to be omitted when computing the determinant.

After changing the time scale according to $\tau \longrightarrow \tau/\sqrt{E_J E_C}$ the bounce action can be rewritten as $S_E[q_b] = 2\sqrt{E_J/E_C}\, s(I/I_c^0)$ and for the inverse decay time we obtain

$$\frac{1}{\tau} \approx 2\frac{(E_J^3 E_C)^{1/4}}{\hbar} K(I/I_c^0) \exp\left(-2\sqrt{\frac{E_J}{E_C}}\, s(I/I_c^0)\right) . \tag{47}$$

Here $K(I/I_c^0)$ is a numerical factor of order one. The function $s(I/I_c^0)$ depending on the only parameter I/I_c^0 can be evaluated numerically by solving the Lagrangian equations for the action (42) with the appropriate boundary conditions. The result is presented on Fig. 4. Let us assume that the current I is not very small so that the energy difference $\delta V = \pi E_J I/I_c^0$ between two nearest minima of the potential (43) is much larger than the quantum amplitude for a phase slip υ introduced above, i.e. we assume that $I \gg I_1 = I_c^0 \upsilon/\pi E_J$. In this case transitions within each rhombus between the states corresponding to different minima of the potential (43) are incoherent. Total voltage along the chain can be expressed in terms of τ as $V = N\hbar\overline{\dot\theta}/2e \approx \pi N\hbar/2e\tau$ since during each jump of the system from one minimum to another the phase θ changes by π. Thus we obtain for low-current $V(I)$ dependence:

$$V(I) = \frac{\pi N E_J}{e}\left(\frac{E_C}{E_J}\right)^{1/4} K(I/I_c^0) \exp\left(-2\sqrt{\frac{E_J}{E_C}}\, s(I/I_c^0)\right) \tag{48}$$

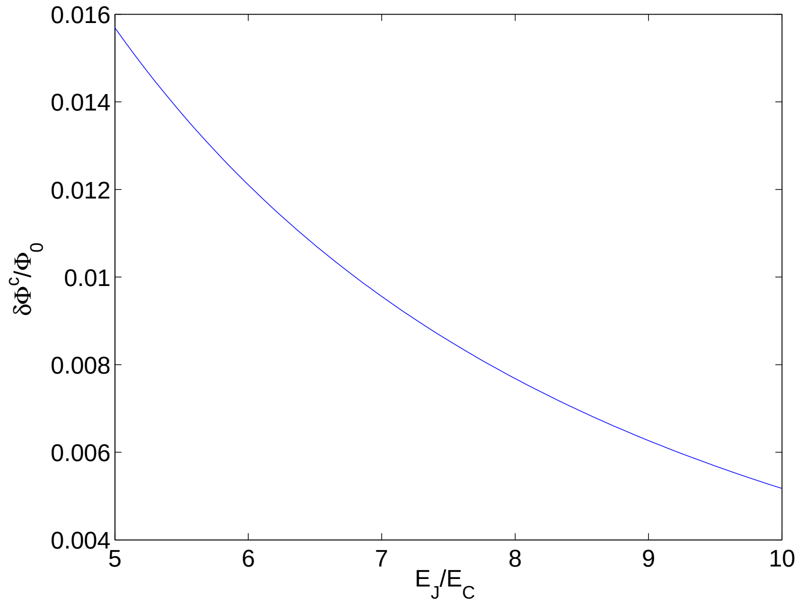

Figure 5: The critical deviation $\delta\Phi^c$ as a function of ratio E_J/E_C

Equation (48) describes the rhombi chain when the bias current I is large enough: $I > I_c$, $I \gg I_1$. Under these conditions coherence in the system is destroyed. This limit is opposite to the one we have considered in previous Section, where the value of equilibrium Josephson current was determined by *coherent* quantum fluctuations of all rhombi.

V. CONCLUSIONS

In this paper we provide detailed calculations of superconductive current in a long chain composed of frustrated rhombi (i.e. loops made of 4 superconductive islands). We show that supercurrent carried in $4e$ quanta dominates over usual $2e$ supercurrent in the close vicinity of the maximally frustrated point $\Phi_r = \Phi_0/2$. According to (41) the critical deviation $\delta\Phi^c$ from this point, which brings the system back to usual $2e$-supercurrent, depends on the only parameter E_J/E_C. This dependence is presented on Fig. 5. We see that $\delta\Phi^c$ rapidly decreases with the increase of the ratio E_J/E_C. In order to observe experimentally the $4e$-supprecurent one should control the flux Φ_r penetrating each rhombus with accuracy better than $\delta\Phi^c$. Thus E_J/E_C should not be too large.

In accordance with (14) the parameter $q = Nw/2$ governing the strength of fluctuations in the maximally frustrated point is proportional to $N^2 v/E_J$ with v defined by equation (9). In

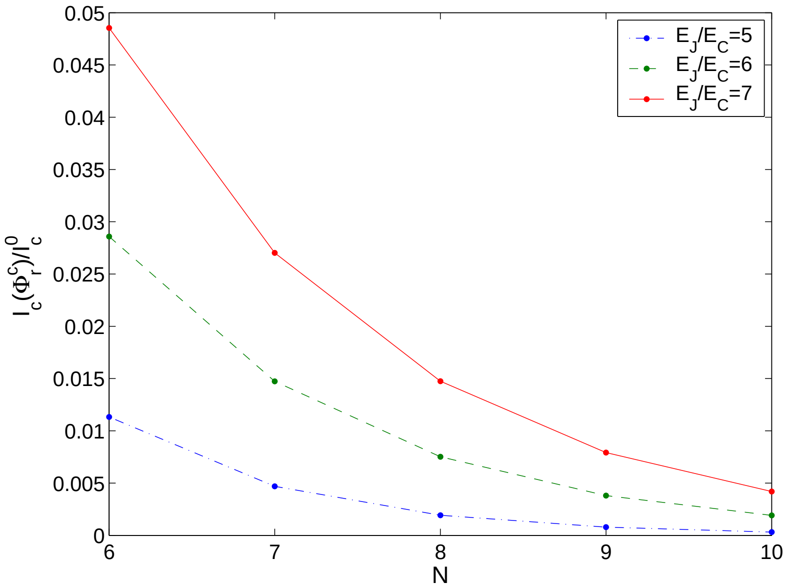

Figure 6: The critical current I_c at the crossover point as a function of N at differnt E_J/E_C.

the regime of strong fluctuations (large q) both $4e$- and $2e$-supercurrents are exponentially small, cf. Eq. (34). The actions S_E^{4e} and S_E^{2e} in (34) are proportional to the number of rhombi N: $S_E^{2e} \sim N\delta\widetilde{S}_E^{2e}$ and $S_E^{4e} \sim N\delta\widetilde{S}_E^{4e}$. At sufficiently large N small variations of $\delta\widetilde{S}_E^{4e}$ and $\delta\widetilde{S}_E^{2e}$ near the crossover point Φ_r^c lead to strong alteration in the relative weight of $2e$- and $4e$-supercurrents. Thus the crossover between $2e$- and $4e$-regimes is expected to be sharp for large N and $N^2\upsilon/E_J \geq 1$. On the other hand, the magnitudes of supercurrent components I_{4e} and I_{2e}, although suppressed by quantum fluctuations, should be not too weak to be measured. The semi-qualitative dependence of the critical current I_c at the crossover point on the number of rhombi at different E_J/E_C is presented on Fig. (6). While calculating the curves depicted on Fig. 6 the pre-exponential factor in the expression for the critical current was evaluated as a geometrical mean of the prefactors in (38) and (40). The optimal set of parameters seems to be the following: $5 \leq E_J/E_C \leq 7$, and $N \in (6, 10)$. According to Figs. 5 and 6 it would give $\delta\Phi^c/\Phi_0 \in (0.01 - 0.015)$ and $I_c \sim 10^{-2} - 10^{-3} I_c^0$.

In this paper we have analysed the path integral (25) for the limit of relatively large δ when $h \gg w$. For this condition to hold at the crossover point we need, cf. (14, 41)

$$\left.\frac{h}{w}\right|_{\Phi_r=\Phi_r^c} = \left(\frac{\pi^2}{64\sqrt{2}d_0^2}\frac{E_J}{\upsilon}\right)^{1/3} \approx 0.2\left(\frac{E_J}{\upsilon}\right)^{1/3} \gg 1 \tag{49}$$

This is always true for large E_J/E_C. However for the proposed set of parameters $(5 \leq E_J/E_C \leq 7)$ the ratio $0.75 \leq h/w \leq 0.95$ and we are at the edge of the validity

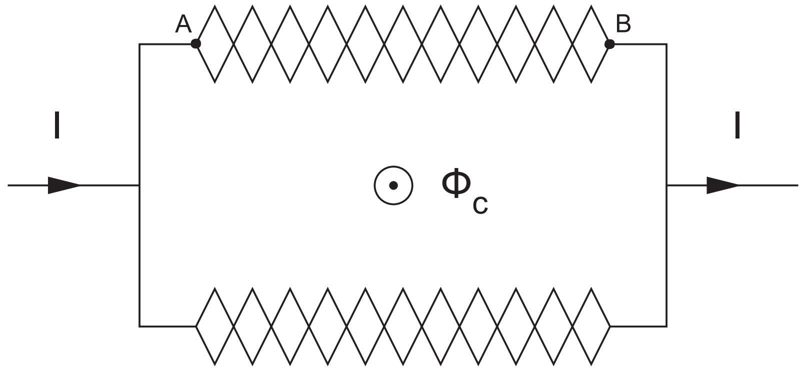

Figure 7: An analog of dc-SQUID configuration for the measurement of the current-phase relation (33).

region for our approximation. Therefore in order to obtain accurate estimates for I_{2e} and I_{4e} at the point of crossover one needs to calculate the classical actions on $2e$- and $4e$-trajectories for the full path integral (25).

Possible experimental arrangement for testing the current-phase relation (33) is presented on Fig. 7. The circuit on Fig. 7 is an analog of a simple dc-SQUID. Let us denote by ϕ the order parameter phase difference in points A and B. It follows from Eq. (33), that in order to evaluate the critical current of the proposed device, much as with a dc-SQUID, one needs to maximize over phase ϕ the current I given by

$$I = 2I_{4e}\cos\frac{2\pi\Phi_c}{\Phi_0}\sin\left(2\phi+\frac{2\pi\Phi_c}{\Phi_0}\right)+2I_{2e}\cos\frac{\pi\Phi_c}{\Phi_0}\sin\left(\phi+\frac{\pi\Phi_c}{\Phi_0}\right). \tag{50}$$

When the deviation $\delta\Phi$ of the magnetic flux Φ_r through each rhombus from $\Phi_0/2$ exceeds the critical deviation $\delta\Phi^c$, the $4e$-supercurrent is negligible and for the critical current of the circuit on Fig. (7) we get (in complete analogy with a dc-SQUID)

$$I_c^s = 2I_{2e}\left|\cos\frac{\pi\Phi_c}{\Phi_0}\right|. \tag{51}$$

So the dependence of I_c^s on the flux Φ^c is Φ_0-periodic for $\delta\Phi \gg \delta\Phi_c$. On the other hand, at the maximally frustrated point only $4e$-supercurrent survives so that I_c^s is $\Phi_0/2$-periodic

$$I_c^s = 2I_{4e}\left|\cos\frac{2\pi\Phi_c}{\Phi_0}\right|. \tag{52}$$

The dependence $I_c^s(\Phi_c)$ at the crossover point ($I_{2e} = I_{4e}$) is presented on Fig. 8.

In our analysis we neglected two intrinsic sources of disorder which are always present in the problem considered: a) some weak randomness of fluxes Φ_r^j penetrating different

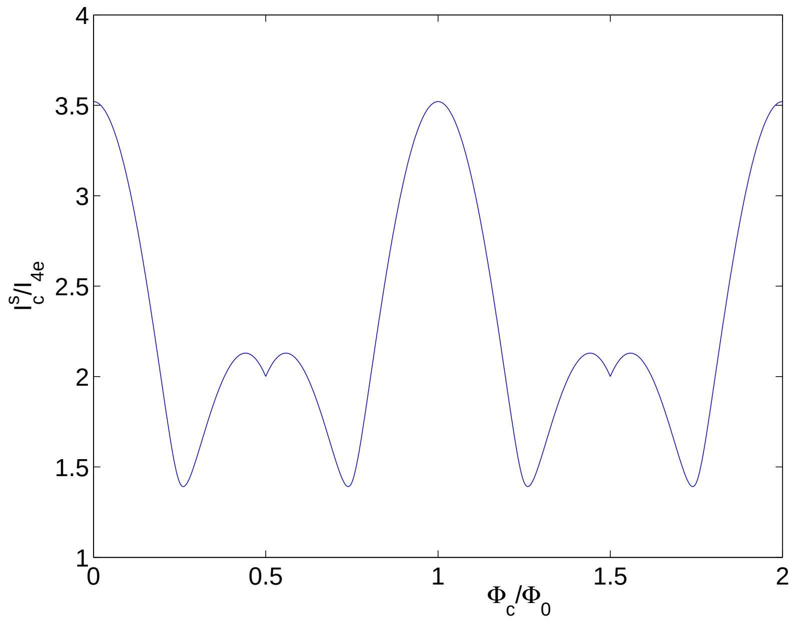

Figure 8: The critical current I_c^s at the crossover point.

rhombi (due to unavoidable differences in their areas), and b) random stray charges q_n which produce, due to Aharonov-Casher effect, some random phase factors to the phase slip tunnelling amplitudes. Whereas the effect of type-a) disorder may be expected to be weak if areas of different rhombi coincide with the accuracy better than $\delta\Phi^c/\Phi_0$, the b)-type effect may occur to be more severe, cf. Ref. [5], where it was discussed for the simple JJ chain. We plan to study these effects in further publications.

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Appendix A: SEMICLASSICAL ANALYSIS FOR $I(\gamma)$

In this Appendix we will obtain the dependence of the lowest eigenvalue b of the problem defined by Eqs.(12,13) on the phase $\tilde{\gamma}$ in the regime of strong fluctuations and derive the expression (33) for the persistent current.

Let us analyse the transition amplitude (26) in more details for the case when (x_1, y_1) is

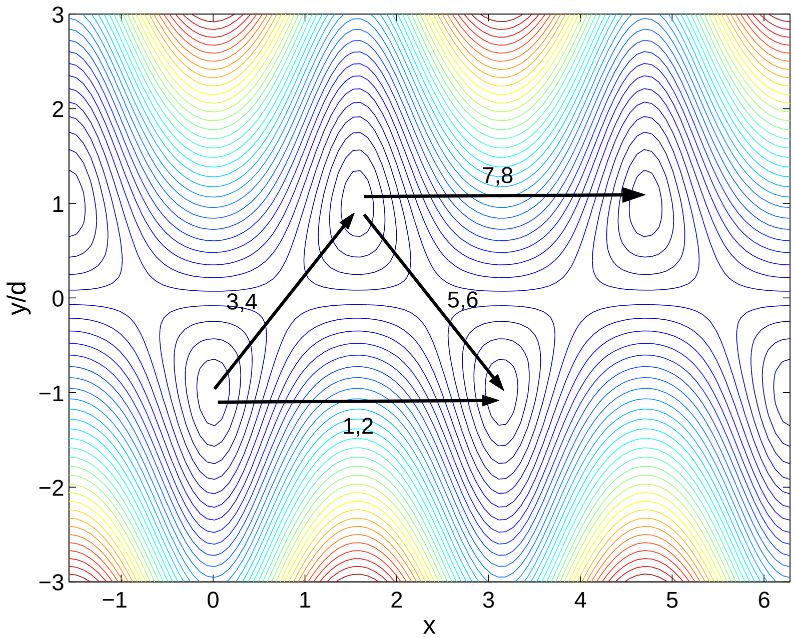

Figure 9: The contour plot of the potential $U_{eff}(x, y)$.

an "even" minimum of the potential (28) and (x_2, y_2) is the neares "odd" one, cf. (31,32). To be specific we choose $(x_1, y_1) = (0, -d)$ and $(x_2, y_2) = (\pi/2, d)$. The contour plot of the potential $U_{eff}(x, y)$ is presented on Fig. 9. Possible tunneling trajectories of the system are schematically depicted with arrows. It is convenient to divide all trajectories into eight groups. Along each trajectory from group 1, variable y is unchanged and equals $-d$ on both ends of the trajectory, whereas variable x inreases by π; a trajectory from group 2 is a counterpart (going against the arrow on Fig. 9) to the previous one. The groups 3, $4, \ldots, 8$ are defined in the same way according to Fig. 9. All trajectories from groups $1, 2, 7, 8$ connects minima of the same parity and so are of the first type according to Section 3, whereas the trajectories from groups $3, 4, 5, 6$ connects minima of opposite parity and are of the second type.

Let us denote by $T\alpha_{4e}$ and $T\alpha_{2e}$ contributions to the tunneling amplitude from a *single* trajectory of the first and the second type respectively, i.e.

$$\alpha_{4e} = \beta_{4e} e^{-S_E^{4e}} \qquad \text{and} \qquad \alpha_{2e} = \beta_{2e} e^{-S_E^{2e}} , \tag{A1}$$

where the prefactors β_{4e} and β_{2e} can be obtained by integration over the fluctuations near the corresponding trajectories. In order to evaluate the transition amplitude (26) in semiclassical approximation one should sum up the contributions from all trajectories consisting of n_1

subtrajectories from group 1, n_2 subtrajectories from group 2 and so on. Such a trajectory including $R = \sum_{k=1}^{8} n_k$ subtrajectories gives to the path integral (26)

$$\frac{T^R}{R!} \alpha_{4e}^{n_3+n_4+n_5+n_6} \alpha_{2e}^{n_1+n_2+n_7+n_8} \tag{A2}$$

As the trajectory under consideration starts at $(0, -d)$ and ends at $(\pi/2, d)$ one should impose two additional constraints upon the integers $n_1, \ldots, n_8$:

$$2(n_1 + n_7) - 2(n_2 + n_8) + n_3 - n_4 + n_5 - n_6 = 1\,, \tag{A3}$$

$$n_3 - n_4 - n_5 + n_6 = 1\,. \tag{A4}$$

Let us introduce

$$K = n_1 + n_7\,, \qquad L = n_2 + n_8 \qquad \text{and} \qquad M = n_4 + n_5 = n_3 + n_6 - 1\,. \tag{A5}$$

All trajectories with fixed K, L, M, n_3 and n_4 give the same contribution to the transition amplitude. The number of such trajectories is

$$\frac{R!(M+1)!M!}{(2M+1)!K!L!n_3!n_4!(M-n_4)!(M-n_3+1)!} \tag{A6}$$

Thus, taking into account (A2,A6,A3), for the transition amplitude in semiclassical approximation we get

$$\langle y_2\,, x_2|\, e^{-T\widehat{H}}\, |y_1\,, x_1\rangle = \sum_{\substack{K,L,n_3,n_4\geq 0\\ M\geq max(n_4,n_3-1)\\ K-L+n_3-n_4=1}} \frac{(T\alpha_{4e})^{K+L}(T\alpha_{2e})^{2M+1}(M+1)!M!}{K!L!n_3!n_4!(2M+1)!(M-n_4)!(M-n_3+1)!} \tag{A7}$$

Instead of calculating the sum in (A7) it is convenient to evaluate function $Q(\alpha_{2e}, \alpha_{4e})$ defined by

$$\langle y_2\,, x_2|\, e^{-T\widehat{H}}\, |y_1\,, x_1\rangle = \frac{1}{T}\frac{\partial Q}{\partial \alpha_{2e}} \tag{A8}$$

For the function $Q(\alpha_{2e}, \alpha_{4e})$ we have

$$Q(\alpha_{2e}, \alpha_{4e}) = \frac{1}{2} \sum_{\substack{K,L,n_3,n_4\geq 0\\ M\geq max(n_4,n_3-1)\\ K-L+n_3-n_4=1}} \frac{(T\alpha_{4e})^{K+L}(T\alpha_{2e})^{2M+2}B(M+1,M+1)}{K!L!n_3!n_4!(M-n_4)!(M-n_3+1)!} \tag{A9}$$

where $B(x, y)$ is Euler's beta function. Using the integral representation for the beta function and the ascending series for the modified Bessel function [7]

$$B(x,y) = 2\int_0^{\pi/2} d\varphi (\sin\varphi)^{2x-1}(\cos\varphi)^{2y-1}\,, \qquad I_n(z) = \left(\frac{1}{2}z\right)^n \sum_k \frac{\left(\frac{1}{4}z^2\right)^k}{k!(n+k)!} \tag{A10}$$

we can carry out the summation over M, K, L in (A9) and obtain

$$Q = \sum_{n_3,n_4\geq 0} \frac{(T\alpha_{2e})^{n_4+n_3+1} I_{|n_4-n_3+1|}(2T\alpha_{4e})}{2^{n_3+n_4+1} n_3! n_4!} \int_0^{\pi} d\varphi (\sin\varphi)^{n_3+n_4} I_{|n_4-n_3+1|}(T\alpha_{2e}\sin\varphi)\,. \quad \text{(A11)}$$

Introducing $Z = n_3 - 1 - n_4$ and accomplishing the summation over n_3, n_4 under fixed Z we get

$$Q = \frac{T\alpha_{2e}}{2} \sum_{Z=-\infty}^{+\infty} \int_0^{\pi} d\varphi I_{|Z|}(2T\alpha_{4e}) I_{|Z|}(T\alpha_{2e}\sin\varphi) I_{|Z+1|}(T\alpha_{2e}\sin\varphi)\,. \quad \text{(A12)}$$

Taking into account the integral representation of the modified Bessel function and its generating function [7]

$$I_n(z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp\left(in\theta + z\cos\theta\right)\,, \qquad \exp\left\{\frac{z}{2}\left(t+\frac{1}{t}\right)\right\} = \sum_{k=-\infty}^{+\infty} t^k I_k(z) \quad \text{(A13)}$$

we can rewrite (A12) in the form

$$Q(\alpha_{2e},\alpha_{4e}) = \frac{T\alpha_{2e}}{4}\int_0^{\pi} d\varphi \int_0^{2\pi} \frac{d\theta_1 d\theta_2}{(2\pi)^2}(\cos\theta_1+\cos\theta_2)\times$$
$$\exp\left\{T\alpha_{2e}\sin\varphi(\cos\theta_1+\cos\theta_2) + 2T\alpha_{4e}\cos(\theta_1+\theta_2)\right\}\,. \quad \text{(A14)}$$

After the substitution $u = (\theta_1+\theta_2)/2$, $v = (\theta_1-\theta_2)/2$ we integrate over φ and v using the relation [12]

$$\int_0^{\pi/2} d\varphi I_1(2z\sin\varphi) = \frac{\pi}{2} I_{1/2}^2(z) = \frac{\sinh^2 z}{z} \quad \text{(A15)}$$

and derive an integral representation for $Q(\alpha_{2e},\alpha_{4e})$

$$Q(\alpha_{2e},\alpha_{4e}) = \int_0^{\pi} \frac{du}{2\pi}\left\{\cosh\left(2T\alpha_{2e}\cos u\right) - 1\right\}\exp\left(2T\alpha_{4e}\cos 2u\right)\,. \quad \text{(A16)}$$

Finally, with the aid of (A8) we obtain an explicit expression for the semiclassical transition amplitude (26)

$$\langle y_2\,,x_2|\, e^{-T\widehat{H}}\,|y_1\,,x_1\rangle = \int_0^{2\pi}\frac{du}{2\pi} e^{iu}\exp(2T\alpha_{4e}\cos 2u + 2T\alpha_{2e}\cos u) \quad \text{(A17)}$$

On the other hand the transition amplitude (26) can be written as a sum over the system's eigenstates

$$\langle y_2\,,x_2|\, e^{-T\widehat{H}}\,|y_1\,,x_1\rangle = \sum_n \psi_n^*(x_1,y_1)\psi_n(x_2,y_2)e^{-E_nT}\,. \quad \text{(A18)}$$

Note that the symmetry group of the potential $U_{eff}(x,y)$ consist of transformations $\widehat{V}_n = \widehat{R}^n\widehat{T}_{\pi n/2}$, where $\widehat{T}_a$ is operator of the translation over distance a along the x-axis introduced in Section 3 and $\widehat{R}$ is operator of the reflection in the x-axis.

Thus the energy levels E_u^0 of a fictitios particle moving in the potential U_{eff} are classified by imposing on their wave functions ψ_u a twisted boundary condition

$$\widehat{V}_1 \psi_u(x,y) \equiv \psi_u(x+\pi/2,-y) = e^{iu}\psi_u(x,y)\,. \tag{A19}$$

Comparing (A18, A19) with (A17) we conclude that the result (A17) has the form of the expansion (A18) with the multiple e^{iu} under the integral emerging from $\psi_u^*(x_1,y_1)\psi_u(x_2,y_2)$ and the remaining part of the under-integral expression providing us with the particle energy

$$E_u^0 = -2\alpha_{4e}\cos 2u - 2\alpha_{2e}\cos u\,. \tag{A20}$$

Coming back to the original problem defined by Eqs. (12, 13) and comparing (12) with (A19) we see that we should identify the phase $\widetilde{\gamma}$ with the "quasimomentum" u. Taking into account the relation between b and the energy of the fictitios particle $b = 2E^0$ mentioned in Section 3 we finally obtain the $b(\widetilde{\gamma})$ dependence:

$$b(\widetilde{\gamma}) = -4\alpha_{4e}\cos 2\widetilde{\gamma} - 4\alpha_{2e}\cos\widetilde{\gamma}\,. \tag{A21}$$

With Eq. (A21) and standard relation $I(\gamma) = (2e/\hbar)dE_0/d\gamma$ we easily recover the results (33) and (34).

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